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EQUILIBRIUM OF A SYSTEM OF BARS USING A MULTIBODY TYPE METHOD

Nicolae-Doru STANESCU[1], Dinel POPA[2]
[1]University of Pitesti, s_doru@yahoo.com
[2]University of Pitesti, dinel_popa@yahoo.com

***Abstract*—**In this paper we purpose a multibody type method to determine the spatial equilibrium of a system of bars, acted only by their own weights. The equilibriums are obtained as the solutions of a system of 18 non-linear equations, which also offers the reactions in the linkages. A numerical example is discussed by two methods: the first one using an optimization procedure to determine the minimum of the potential energy and the second one using the technique proposed in this paper, the results being identical.

I. Introduction

The multibody methods are a new approach in treating the dynamics of systems of rigid bodies. Weber and Arnold [1] consider a planar mobile robot schematized a quadrilateral mechanism with variable lengths, which is studied with the aid of the perturbation theory. Bai and Zhao [2] studied multibody systems with revolute clearance joints and introduced indicators for qualitative analysis. Nikravesh and Gim [3] used the joint coordinate formulation for automatic generation of the equations of motion and applied it to the dynamical behavior of a sport car simulation. Lan *et al*. [4] used the generalized multiple shooting method for the analysis of the dynamics of elastic mechanisms. Bai *et al*. [5] discussed the planar mechanisms with revolute clearance joints wear. Hwang [6] described a method for the closed-loop flexible mechanical systems and it used it at a quadrilateral mechanism. McPhee [7] used the graph theory in the study of a torque-driven planar four-bar mechanism. Bayo and Avello [8] used Lagrangian algorithms for the study of the constrained multibody dynamics and applied the theory to a slider-crank mechanism. Malczyk and Fraczek [9] treated tree topology mechanisms or multibody systems which contain kinematic closed loops with applications to a double loop mechanism. Haddadpour *et al*. [10] performed a comparison between genetic algorithms and optimization techniques for the analysis of equilibrium and discussed a planar mechanism with six elements and a spatial slider-crank mechanism.

II. Mathematical Model

We consider the system captured in Fig. 1., consisting in the bars OA_1, A_1A_2, and A_2A_3, respectively, of lengths l_1, l_2, and l_3, respectively. The positions of the centers of weight are given by the dimensions $OC_1 = r_1$, $A_1C_2 = r_2$, and $A_2C_3 = r_3$. The ends O and A_3 of the first and the third bar are fixed, and they have the coordinates 0, 0, 0, and X_3, Y_3, Z_3, respectively, with respect to the fixed reference system $OXYZ$.

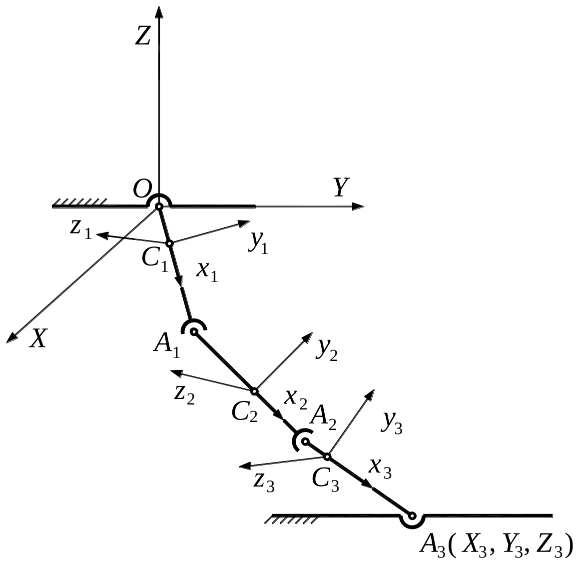

Fig. 1. Mechanical model.

One knows the masses m_1, m_2 and m_3 of the three bars. For each bar we choose the system of the principal central axes of inertia $C_ix_iy_iz_i$, $i=\overline{1,3}$. The principal central moments of inertia J_{x_i}, J_{y_i}, J_{z_i}, $i=\overline{1,3}$, are also known.

The position of an arbitrary bar is defined by the coordinates X_{C_i}, Y_{C_i}, Z_{C_i} of its center of weight, and by the Bryan rotational angles ψ_i, θ_i, φ_i, $i=\overline{1,3}$.

In this way, the position of the system is described by 18 parameters, not all of them being independents.

The bars are acted only by their own weights.

III. THE CONSTRAINT FUNCTIONS

Let $[\mathbf{A}_i]$, $i=\overline{1,3}$, the rotational matrices for the three bars. We may write

$$[\mathbf{A}_i]=[\boldsymbol{\psi}_i][\boldsymbol{\theta}_i][\boldsymbol{\varphi}_i],\ i=\overline{1,3}, \tag{1}$$

in which

$$[\boldsymbol{\psi}_i]=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\psi_i & -\sin\psi_i\\ 0 & \sin\psi_i & \cos\psi_i\end{bmatrix}, \tag{2}$$

$$[\boldsymbol{\theta}_i]=\begin{bmatrix}\cos\theta_i & 0 & \sin\theta_i\\ 0 & 1 & 0\\ -\sin\theta_i & 0 & \cos\theta_i\end{bmatrix}, \tag{3}$$

$$[\boldsymbol{\varphi}_i]=\begin{bmatrix}\cos\varphi_i & -\sin\varphi_i & 0\\ \sin\varphi_i & \cos\varphi_i & 0\\ 0 & 0 & 1\end{bmatrix},\ i=\overline{1,3}. \tag{4}$$

It results

$$[\mathbf{A}_i]=\begin{bmatrix}\mathrm{c}\theta_i\mathrm{c}\varphi_i & -\mathrm{c}\theta_i\mathrm{s}\varphi_i & \mathrm{s}\theta_i\\ \begin{matrix}\mathrm{s}\psi_i\mathrm{s}\theta_i\mathrm{c}\varphi_i\\ +\mathrm{c}\psi_i\mathrm{s}\varphi_i\end{matrix} & \begin{matrix}-\mathrm{s}\psi_i\mathrm{s}\theta_i\mathrm{s}\varphi_i\\ +\mathrm{c}\psi_i\mathrm{c}\varphi_i\end{matrix} & -\mathrm{s}\psi_i\mathrm{c}\theta_i\\ \begin{matrix}-\mathrm{c}\psi_i\mathrm{s}\theta_i\mathrm{c}\varphi_i\\ +\mathrm{s}\psi_i\mathrm{s}\theta_i\end{matrix} & \begin{matrix}\mathrm{c}\psi_i\mathrm{s}\theta_i\mathrm{s}\varphi_i\\ +\mathrm{s}\psi_i\mathrm{c}\varphi_i\end{matrix} & \mathrm{c}\psi_i\mathrm{c}\varphi_i\end{bmatrix},\ i=\overline{1,3}. \tag{5}$$

We may write

$$\begin{bmatrix}X_O\\ Y_O\\ Z_O\end{bmatrix}=\begin{bmatrix}X_{C_1}\\ Y_{C_1}\\ Z_{C_1}\end{bmatrix}+[\mathbf{A}_1]\begin{bmatrix}-r_1\\ 0\\ 0\end{bmatrix}, \tag{6}$$

wherefrom one gets the constraint functions

$$f_1=X_{C_1}-r_1\cos\theta_1\cos\varphi_1=0, \tag{7}$$

$$\begin{gathered}f_2=Y_{C_1}\\ -r_1(\sin\psi_1\sin\theta_1\cos\varphi_1+\cos\psi_1\sin\varphi_1)=0,\end{gathered} \tag{8}$$

$$\begin{gathered}f_3=Z_{C_1}\\ -r_1(-\cos\psi_1\sin\theta_1\cos\varphi_1+\sin\psi_1\sin\varphi_1)=0.\end{gathered} \tag{9}$$

The coordinates of the point A_1 are

$$\begin{bmatrix}X_{A_1}\\ Y_{A_1}\\ Z_{A_1}\end{bmatrix}=\begin{bmatrix}X_{C_1}\\ Y_{C_1}\\ Z_{C_1}\end{bmatrix}+[\mathbf{A}_1]\begin{bmatrix}l_1-r_1\\ 0\\ 0\end{bmatrix} \tag{10}$$

and

$$\begin{bmatrix}X_{A_1}\\ Y_{A_1}\\ Z_{A_1}\end{bmatrix}=\begin{bmatrix}X_{C_2}\\ Y_{C_2}\\ Z_{C_2}\end{bmatrix}+[\mathbf{A}_2]\begin{bmatrix}-r_2\\ 0\\ 0\end{bmatrix}, \tag{11}$$

respectively, wherefrom, by equating, one gets the constraint functions

$$\begin{gathered}f_4=X_{C_1}+(l_1-r_1)\cos\theta_1\cos\varphi_1\\ -X_{C_2}+r_2\cos\theta_2\cos\varphi_2=0,\end{gathered} \tag{12}$$

$$\begin{gathered}f_5=Y_{C_1}+(l_1-r_1)(\sin\psi_1\sin\theta_1\cos\varphi_1\\ +\cos\psi_1\sin\varphi_1)-Y_{C_2}\\ +r_2(\sin\psi_2\sin\theta_2\cos\varphi_2+\cos\psi_2\sin\varphi_2)=0,\end{gathered} \tag{13}$$

$$\begin{gathered}f_6=Z_{C_1}+(l_1-r_1)(-\cos\psi_1\sin\theta_1\cos\varphi_1\\ +\sin\psi_1\sin\varphi_1)-Z_{C_2}\\ +r_2(-\cos\psi_2\sin\theta_2\cos\varphi_2+\sin\psi_2\sin\varphi_2)=0.\end{gathered} \tag{14}$$

Analogically, for the point A_2 we find the constraint functions

$$\begin{gathered}f_7=X_{C_2}+(l_2-r_2)\cos\theta_2\cos\varphi_2\\ -X_{C_3}+r_3\cos\theta_3\cos\varphi_3=0,\end{gathered} \tag{15}$$

$$\begin{gathered}f_8=Y_{C_2}+(l_2-r_2)(\sin\psi_2\sin\theta_2\cos\varphi_2\\ +\cos\psi_2\sin\varphi_2)-Y_{C_3}\\ +r_3(\sin\psi_3\sin\theta_3\cos\varphi_3+\cos\psi_3\sin\varphi_3)=0,\end{gathered} \tag{16}$$

$$\begin{gathered}f_9=Z_{C_2}+(l_2-r_2)(-\cos\psi_2\sin\theta_2\cos\varphi_2\\ +\sin\psi_2\sin\varphi_2)-Z_{C_3}\\ +r_3(-\cos\psi_3\sin\theta_3\cos\varphi_3+\sin\psi_3\sin\varphi_3)=0,\end{gathered} \tag{17}$$

while for the point A_3 we have the constraint functions

$$f_{10}=X_{C_2}+(l_2-r_2)\cos\theta_2\cos\varphi_2-X_3=0, \tag{18}$$

$$\begin{gathered}f_{11}=Y_{C_2}+(l_2-r_2)(\sin\psi_2\sin\theta_2\cos\varphi_2\\ +\cos\psi_2\sin\varphi_2)-Y_3=0,\end{gathered} \tag{19}$$

$$f_{12} = Z_{C_2} + (l_2 - r_2)(-\cos\psi_2 \sin\theta_2 \cos\varphi_2 + \sin\psi_2 \sin\varphi_2) - Z_3 = 0. \tag{20}$$

In this way, it results that the system has only 6 degrees of freedom, three of them being useless, name them, the rotations about the axes $C_i x_i$, $i = \overline{1,3}$.

IV. The Matrix of Constraints

We denote by $\{\mathbf{q}\}$ the vector

$$\{\mathbf{q}\} = [X_{C_1} \ \ldots \ \varphi_1 \ \ldots \ X_{C_3} \ \ldots \ \varphi_3]^{\mathrm{T}}, \tag{21}$$

the functions f_i, $i = \overline{1,12}$, depending on $\{\mathbf{q}\}$.

The entries $b_{i,j}$ of the matrix of constraints $[\mathbf{B}]$ are given by

$$b_{i,j} = \frac{\partial f_i}{\partial q_j}, \ i = \overline{1,12}, \ j = \overline{1,18}. \tag{22}$$

V. The Inertial Matrices

Using the notations in [11], [12], we have

$$[\mathbf{Q}_i] = [\boldsymbol{\varphi}_i]^{\mathrm{T}}\left[[\boldsymbol{\theta}_i]^{\mathrm{T}}\{\mathbf{u}_\psi\} \ \{\mathbf{u}_\theta\} \ \{\mathbf{u}_\varphi\}\right] = \begin{bmatrix} \cos\theta_i \cos\varphi_i & \sin\varphi_i & 0 \\ -\cos\theta_i \sin\varphi_i & \cos\varphi_i & 0 \\ \sin\theta_i & 0 & 1 \end{bmatrix}, \quad i = \overline{1,3}, \tag{23}$$

$$\{\boldsymbol{\omega}_i\} = [\mathbf{Q}_i]\begin{bmatrix} \dot\psi_i \\ \dot\theta_i \\ \dot\varphi_i \end{bmatrix} = \begin{bmatrix} -\dot\psi_i \cos\theta_i \cos\varphi_i + \dot\theta_i \sin\varphi_i \\ -\dot\psi_i \cos\theta_i \sin\varphi_i + \dot\theta_i \cos\varphi_i \\ \dot\psi_i \sin\theta_i + \dot\varphi_i \end{bmatrix} = \begin{bmatrix} \omega_{x_i} \\ \omega_{y_i} \\ \omega_{z_i} \end{bmatrix}, \quad i = \overline{1,3}, \tag{24}$$

$$[\boldsymbol{\omega}_i] = \begin{bmatrix} 0 & -\omega_{z_i} & \omega_{y_i} \\ \omega_{z_i} & 0 & -\omega_{x_i} \\ -\omega_{y_i} & \omega_{x_i} & 0 \end{bmatrix}, \ i = \overline{1,3}, \tag{25}$$

$$[\mathbf{J}_{C_i}] = \begin{bmatrix} J_{x_i} & 0 & 0 \\ 0 & J_{y_i} & 0 \\ 0 & 0 & J_{z_i} \end{bmatrix}, \ i = \overline{1,3}, \tag{26}$$

where we kept into account that the axes $C_i x_i$, $C_i y_i$, and $C_i z_i$, $i = \overline{1,3}$, are principal central axes of inertia.

Since

$$x_{C_i} = y_{C_i} = z_{C_i} = 0, \ i = \overline{1,3}, \tag{27}$$

we deduce

$$[\mathbf{S}_i] = \begin{bmatrix} 0 & -mz_{C_i} & my_{C_i} \\ mz_{C_i} & 0 & -mx_{C_i} \\ -my_{C_i} & mx_{C_i} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ i = \overline{1,3}, \tag{28}$$

wherefrom it results

$$[\mathbf{A}_i][\mathbf{S}_i]^{\mathrm{T}}[\mathbf{Q}_i] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ i = \overline{1,3}, \tag{29}$$

$$[\mathbf{Q}_i]^{\mathrm{T}}[\mathbf{S}_i][\mathbf{A}_i]^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ i = \overline{1,3}. \tag{30}$$

We find

$$[\mathbf{Q}_i]^{\mathrm{T}}[\mathbf{J}_{C_i}][\mathbf{Q}_i] = \begin{bmatrix} J_{x_i}\mathrm{c}^2\varphi_i\mathrm{c}^2\theta_i + J_{y_i}\mathrm{s}^2\varphi_i\mathrm{c}^2\theta_i + J_{z_i}\mathrm{s}^2\theta_i & J_{x_i}\mathrm{c}\varphi_i\mathrm{c}\theta_i\mathrm{s}\varphi_i - J_{y_i}\mathrm{c}\varphi_i\mathrm{c}\theta_i\mathrm{s}\varphi_i & J_{z_i}\mathrm{s}\theta_i \\ J_{x_i}\mathrm{c}\varphi_i\mathrm{s}\theta_i\mathrm{s}\varphi_i - J_{x_i}\mathrm{c}\varphi_i\mathrm{s}\theta_i\mathrm{s}\varphi_i & J_{x_i}\mathrm{s}^2\varphi_i + J_{y_i}\mathrm{c}^2\varphi_i & 0 \\ J_{z_i}\mathrm{s}\varphi_i & 0 & J_{z_i} \end{bmatrix}, \quad i = \overline{1,3}, \tag{31}$$

and the matrices of inertia

$$[\mathbf{M}_i] = \begin{bmatrix} \begin{bmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & m_i \end{bmatrix} & [\mathbf{0}] \\ [\mathbf{0}] & [\mathbf{Q}_i]^{\mathrm{T}}[\mathbf{J}_{C_i}][\mathbf{Q}_i] \end{bmatrix}, \ i = \overline{1,3}. \tag{32}$$

VI. The Equation of Motion

This equation reads

$$\begin{bmatrix} [\mathbf{M}_1] & [\mathbf{0}_{6,6}] & [\mathbf{0}_{6,6}] & \\ [\mathbf{0}_{6,6}] & [\mathbf{M}_2] & [\mathbf{0}_{6,6}] & -[\mathbf{B}]^{\mathrm{T}} \\ [\mathbf{0}_{6,6}] & [\mathbf{0}_{6,6}] & [\mathbf{M}_3] & \\ & [\mathbf{B}] & & [\mathbf{0}_{12,12}] \end{bmatrix} \begin{bmatrix} \{\ddot{\mathbf{q}}\} \\ \{\boldsymbol{\lambda}\} \end{bmatrix} = \begin{bmatrix} \{\mathbf{F}_{\mathbf{q}_1}\} + \{\tilde{\mathbf{F}}_{\mathbf{q}_1}\} \\ \{\mathbf{F}_{\mathbf{q}_2}\} + \{\tilde{\mathbf{F}}_{\mathbf{q}_2}\} \\ \{\mathbf{F}_{\mathbf{q}_3}\} + \{\tilde{\mathbf{F}}_{\mathbf{q}_3}\} \\ -[\dot{\mathbf{B}}]\{\dot{\mathbf{q}}\} \end{bmatrix}, \tag{33}$$

where

$$\{\mathbf{F}_{\mathbf{q}_i}\} = \begin{bmatrix} \{\mathbf{F}_{\mathbf{s}_i}\} \\ \{\mathbf{F}_{\boldsymbol{\beta}_i}\} \end{bmatrix},\ i = \overline{1,3}, \tag{34}$$

$$\{\mathbf{F}_{\mathbf{s}_i}\} = [0\ \ 0\ \ -m_i g]^{\mathrm{T}},\ i = \overline{1,3}, \tag{35}$$

$$\{\mathbf{F}_{\boldsymbol{\beta}_i}\} = [0\ \ 0\ \ 0]^{\mathrm{T}},\ i = \overline{1,3}, \tag{36}$$

$$\{\tilde{\mathbf{F}}_{\mathbf{q}_i}\} = \begin{bmatrix} \{\tilde{\mathbf{F}}_{\mathbf{s}_i}\} \\ \{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_i}\} \end{bmatrix},\ i = \overline{1,3}, \tag{37}$$

$$\{\tilde{\mathbf{F}}_{\mathbf{s}_i}\} = -\left[[\mathbf{A}_i][\mathbf{S}_i]^{\mathrm{T}}[\dot{\mathbf{Q}}_i] + [\dot{\mathbf{A}}_i][\mathbf{S}_i]^{\mathrm{T}}[\mathbf{Q}_i]\right]$$
$$\begin{bmatrix} \dot{\psi}_i \\ \dot{\theta}_i \\ \dot{\varphi}_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\quad i = \overline{1,3}, \tag{38}$$

$$\{\tilde{\mathbf{F}}_{\boldsymbol{\beta}_i}\} = -\left[[\mathbf{Q}_i]^{\mathrm{T}}[\mathbf{J}_{C_i}][\dot{\mathbf{Q}}_i]\right.$$
$$\left. + [\dot{\mathbf{Q}}_i]^{\mathrm{T}}[\boldsymbol{\omega}_i][\mathbf{J}_{C_i}][\mathbf{Q}_i]\right] \begin{bmatrix} \dot{\psi}_i \\ \dot{\theta}_i \\ \dot{\varphi}_i \end{bmatrix},\quad i = \overline{1,3}, \tag{39}$$

while $[\mathbf{0}_{r,c}]$ is the null matrix with r rows and c columns.

In addition, $\{\boldsymbol{\lambda}\}$ defines the matrix of Lagrange's multipliers; it has 12 rows and 1 column.

VII. The Equilibrium Positions

At an equilibrium position, the parameters X_{C_i}, Y_{C_i}, Z_{C_i}, ψ_i, θ_i, φ_i, $i = \overline{1,3}$, have constant values; hence all their derivatives with respect to time vanish.

The moving equation (45) reads now

$$-[\mathbf{B}]^{\mathrm{T}}\{\boldsymbol{\lambda}\} = [\mathbf{T}], \tag{40}$$

in which the matrix $[\mathbf{T}]$ has the expression

$$[\mathbf{T}] = [0\ \ 0\ \ -m_1 g\ \ 0\ \ \ldots\ \ 0\ \ -m_3 g\ \ 0\ \ 0\ \ 0]^{\mathrm{T}}. \tag{41}$$

The relation (52) describes a system of 18 equations 21 unknowns: 12 parameters λ_i and 9 angles ψ_1, θ_1, φ_1, …, ψ_3, θ_3, φ_3. At this system we add the 12 functions of constrain resulting a system of 30 equations with 30 unknowns: 12 parameters λ_i, and the values X_{C_1}, Y_{C_1}, …, φ_3.

VIII. Example

Let us consider the system of three bars in Fig. 2., where the dimensions are $OA = l$, $AB = l$, $BD = l$. The points O and D are situated on the same vertical, and the bars are homogeneous and of same mass m.

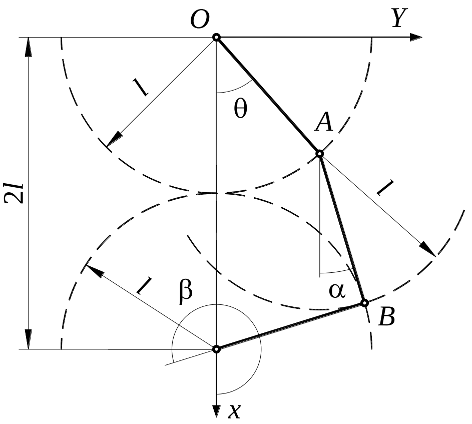

Fig. 2. Example.

We denote by θ, α, and β the angles between the three bars and the vertical descendent bar Ox. One knows the distance $OD = l$.

One asks the equilibrium positions.

Solution 1: We may write the relations

$$l\cos\theta + l\cos\alpha + l\cos\beta = 2l, \tag{i}$$

$$l\sin\theta + l\sin\alpha + l\sin\beta = 0, \tag{ii}$$

wherefrom we get

$$\cos\theta + \cos\alpha + \cos\beta = 2, \tag{iii}$$

$$\sin\theta + \sin\alpha + \sin\beta = 0. \tag{iv}$$

The potential energy of the system reads

$$V = mgl(6 - 2.5\cos\theta - 1.5\cos\alpha - 0.5\cos\beta). \tag{v}$$

Keeping into account the expressions (iii), it results

$$V = mgl(5 - 2\cos\theta - \cos\alpha). \tag{vi}$$

Writing the expressions (iii) and (iv) as

$$\cos\beta = 2 - \cos\theta - \cos\alpha\,,\ \sin\beta = -\sin\theta - \sin\alpha\,, \qquad \text{(viii)}$$

by squaring and summing, we get

$$\begin{aligned}&(20-16\cos\theta)\cos^2\alpha\\ &+2\left(26\cos\theta-20-8\cos^2\theta\right)\cos\alpha\\ &+16\cos^2\theta+25-40\cos\theta-4\sin^2\theta=0\,.\end{aligned} \qquad \text{(viii)}$$

Equation (viii) has two solutions, corresponding to α_1 in the first quadrant, and to α_2 in the fourth quadrant.

Considering the angle θ as variable, from the equation (viii) one finds the angle α and then, with the aid of the expression (vi), one determines the potential energy. The equilibrium positions correspond to the local minimums of the potential energy.

One finds the values:

$$\begin{aligned}&\theta = 19.70975^0\,,\ \alpha = 38.58306^0\,,\\ &\beta = 286.07420^0\,,\ V = 2.33547mgl\,,\end{aligned} \qquad \text{(ix)}$$

and

$$\begin{aligned}&\theta = 0.05730^0\,,\ \alpha = 299.94274^0\,,\\ &\beta = 59.94267^0\,,\ V = 2.50087mgl\,,\end{aligned} \qquad \text{(x)}$$

respectively.

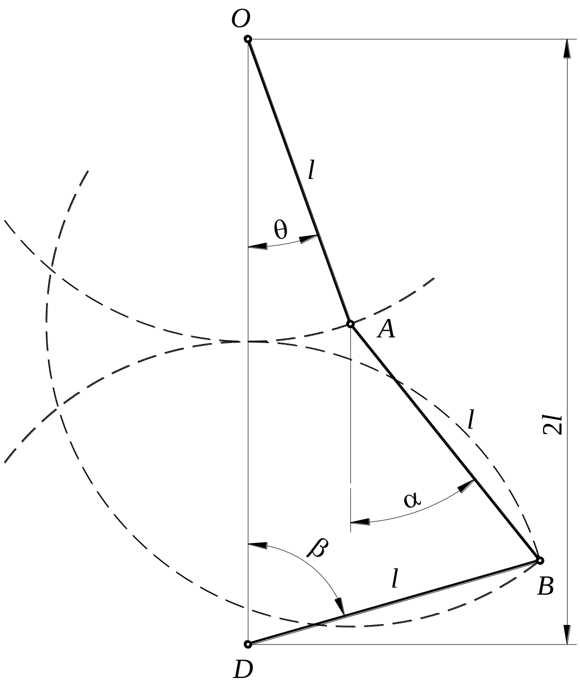

Fig. 3. First equilibrium position.

The equilibrium positions are captured in Fig. 3., for the first case, and Fig. 4., for the second one. Of course, the other two equilibrium positions are given by the symmetric ones with respect to OD.

Solution 2: We will use now the expression (40). We will search only the equilibrium positions in the plane OYZ. It results that the angles ψ_i, $i=\overline{1,3}$, are arbitrary and we will choose $\psi_i = 0$, $i=\overline{1,3}$. Similarly, we deduce the values $\theta_i = \frac{\pi}{2}$, $i=\overline{1,3}$. Keeping into account these expressions, the components of the matrix of constraints are

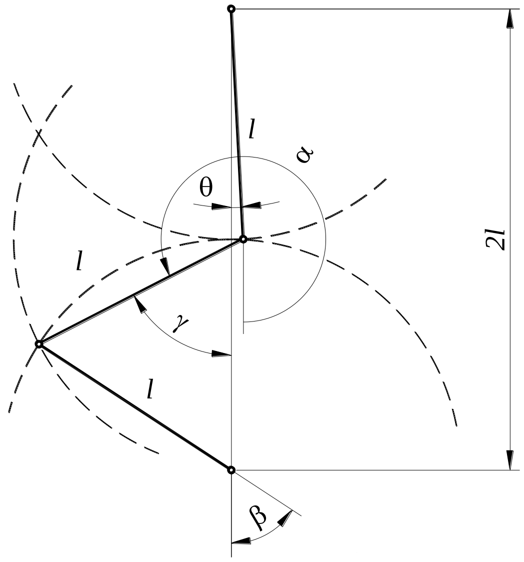

Fig. 4. Second equilibrium position.

$$b_{1,1}=1\,,\ b_{1,5}=\frac{l}{2}\cos\varphi_1\,, \qquad \text{(xi)}$$

$$b_{2,2}=1\,,\ b_{2,4}=-\frac{l}{2}\cos\varphi_1\,,\ b_{2,6}=-\frac{l}{2}\cos\varphi_1 \qquad \text{(xii)}$$

$$b_{3,3}=1\,,\ b_{3,4}=-\frac{l}{2}\sin\varphi_1\,,\ b_{3,6}=-\frac{l}{2}\sin\varphi_1\,, \qquad \text{(xiii)}$$

$$\begin{aligned}&b_{4,1}=1\,,\ b_{4,5}=-\frac{l}{2}\cos\varphi_1\,,\ b_{4,7}=-1\,,\\ &b_{4,11}=-\frac{l}{2}\cos\varphi_2\,,\end{aligned} \qquad \text{(xiv)}$$

$$\begin{aligned}&b_{5,2}=1\,,\ b_{5,4}=\frac{l}{2}\cos\varphi_1\,,\ b_{5,6}=\frac{l}{2}\cos\varphi_1\,,\ b_{5,8}=-1\,,\\ &b_{5,10}=\frac{l}{2}\cos\varphi_2\,,\ b_{5,12}=\frac{l}{2}\cos\varphi_2\,,\end{aligned} \qquad \text{(xv)}$$

$$b_{6,3}=1\,,\ b_{6,4}=\frac{l}{2}\sin\varphi_1\,,\ b_{6,6}=\frac{l}{2}\sin\varphi_1\,,\ b_{6,9}=-1\,,$$

$$b_{6,10} = \frac{l}{2}\sin\varphi_2 ,\ b_{6,12} = \frac{l}{2}\sin\varphi_2 , \tag{xvi}$$

$$b_{7,7} = 1,\ b_{7,11} = -\frac{l}{2}\cos\varphi_2 ,\ b_{7,13} = -1,$$
$$b_{7,17} = -\frac{l}{2}\cos\varphi_3 , \tag{xvii}$$

$$b_{8,8} = 1,\ b_{8,10} = \frac{l}{2}\cos\varphi_2 ,\ b_{8,12} = \frac{l}{2}\cos\varphi_2 ,\ b_{8,14} = -1,$$
$$b_{8,16} = \frac{l}{2}\cos\varphi_3 ,\ b_{8,18} = \frac{l}{2}\cos\varphi_3 , \tag{xviii}$$

$$b_{9,9} = 1,\ b_{9,10} = \frac{l}{2}\sin\varphi_2 ,\ b_{9,12} = \frac{l}{2}\sin\varphi_2 ,\ b_{9,15} = -1,$$
$$b_{9,16} = \frac{l}{2}\sin\varphi_3 ,\ b_{9,18} = \frac{l}{2}\sin\varphi_3 , \tag{xix}$$

$$b_{10,13} = 1,\ b_{10,17} = -\frac{l}{2}\cos\varphi_3 , \tag{xx}$$

$$b_{11,14} = 1,\ b_{11,16} = \frac{l}{2}\cos\varphi_3 ,\ b_{11,18} = \frac{l}{2}\cos\varphi_3 \tag{xxi}$$

$$b_{12,15} = 1,\ b_{12,16} = \frac{l}{2}\sin\varphi_3 ,\ b_{12,18} = \frac{l}{2}\sin\varphi_3 , \tag{xxii}$$

One obtains the system of equations

$$\begin{gathered}
-\lambda_1 - \lambda_4 = 0,\ -\lambda_2 - \lambda_5 = 0,\ -\lambda_3 - \lambda_6 = -mg ,\\
-b_{2,4}\lambda_2 - b_{3,4}\lambda_3 - b_{5,4}\lambda_5 - b_{6,4}\lambda_6 = 0,\ -b_{4,5}\lambda_4 = 0,\\
-b_{2,6}\lambda_2 - b_{3,6}\lambda_3 - b_{5,6}\lambda_5 - b_{6,6}\lambda_6 = 0,\ \lambda_4 - \lambda_7 = 0,\\
\lambda_5 - \lambda_8 = 0,\ \lambda_6 - \lambda_9 = -mg ,\\
-b_{5,10}\lambda_5 - b_{6,10}\lambda_6 - b_{8,10}\lambda_8 - b_{9,10}\lambda_9 = 0,\\
-b_{4,11}\lambda_4 - b_{7,11}\lambda_7 = 0,\\
-b_{5,12}\lambda_5 - b_{6,12}\lambda_6 - b_{8,12}\lambda_8 - b_{9,12}\lambda_9 = 0,\\
\lambda_7 - \lambda_{10} = 0,\ \lambda_8 - \lambda_{11} = 0,\ \lambda_9 - \lambda_{12} = -mg ,\\
-b_{8,16}\lambda_8 - b_{9,16}\lambda_9 - b_{11,16}\lambda_{11} - b_{12,16}\lambda_{12} = 0,\\
-b_{7,17}\lambda_7 - b_{10,17}\lambda_{10} = 0,\\
-b_{8,18}\lambda_8 - b_{9,18}\lambda_9 - b_{11,18}\lambda_{11} - b_{12,18}\lambda_{12} = 0.
\end{gathered} \tag{xxiii}$$

Since $\cos\varphi_1 \neq 0$ (see Solution 1 where we proved that $\cos\theta \neq 0$, it results $b_{4,5} \neq 0$, and the values $\lambda_1 = 0$, $\lambda_4 = 0$, $\lambda_7 = 0$, $\lambda_{10} = 0$.

The system reads now

$$\begin{gathered}
\lambda_2 + \lambda_5 = 0,\ \lambda_3 + \lambda_6 = mg ,\\
\lambda_2\cos\varphi_1 + \lambda_3\sin\varphi_1 - \lambda_5\cos\varphi_1 - \lambda_6\sin\varphi_1 = 0,\\
\lambda_5 - \lambda_8 = 0,\ \lambda_6 - \lambda_9 = -mg ,\\
\lambda_5\cos\varphi_2 - \lambda_6\sin\varphi_2 - \lambda_8\cos\varphi_2 - \lambda_9\sin\varphi_2 = 0,\\
\lambda_8 - \lambda_{11} = 0,\ \lambda_9 - \lambda_{12} = -mg ,\\
-\lambda_8\cos\varphi_3 - \lambda_9\sin\varphi_3\\
-\lambda_{11}\cos\varphi_3 - \lambda_{12}\sin\varphi_3 = 0.
\end{gathered} \tag{xxiv}$$

At these equations we add the relations (iii) and (iv) written as

$$\cos\varphi_1 + \cos\varphi_2 + \cos\varphi_3 - 2 = 0, \tag{xxv}$$

$$\sin\varphi_1 + \sin\varphi_2 + \sin\varphi_3 = 0. \tag{xxvi}$$

The reader can easily obtain the same equilibrium position as in the Solution 1. In addition, the parameters λ_i are the values of the reactions at the points O, A, B, and D.

IX. Conclusions

Our paper presented a new multibody type method for the determination of the equilibrium positions. This method leads to a system of nonlinear equations having as unknown the parameters that define the equilibriums and the reactions in the linkages. The system is solved by numerical methods, the obtained equilibrium position depending on the initial conditions chosen for the first step of iteration. In our example we limited ourselves to the planar equilibrium positions. Obviously, all the positions obtained from one equilibrium positions by rotation about the vertical axes are also equilibrium positions. It results that our system has at least a double uncountable infinity of equilibrium positions.

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