ANNALS OF THE ORADEA UNIVERSITY
Fascicle of Management and Technological Engineering
ISSUE #2, SEPTEMBER 2013, http://www.imtuoradea.ro/auo.fmte/

MULTIPLICITY AS AN OBJECT OF STUDY IN CONTEMPORARY DIFFERENTIAL GEOMETRY

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***Abstract*—**The study presents introductory concepts of multiplicity as one of the basic concepts of differential topology. The work gives the definition of Hausdorff space and some theorems concerning Hausdorff space.

***Keywords*—**Multiplicity, topological space, local map, smooth atlas.

I. Introduction

FROM classical differential geometry, elementary pads S^2 in R^2 as well as pads S^2 and T^2 of the space E^3 are well known. Mentioned pads are supplied with metrics, i.e. scalar product in tangent space, in every spot of that space.

In this paper, we will introduce the term *multiplicity* which can be perceived as *n*-dimensional generalization of curves and pads of space R^n, which is the object of study in contemporary differential geometry.

Definition 1. *Homomorphism φ of open set U from M on open set $\varphi(U)$, of the space R^n, we call p-dimensional local map or local system of coordinates, on topological space M. Set U is called the local map area and it is marked with (U, φ).*

For example, pad S of the space R^3, along with topology of subspace, is a topological subspace. If (V,r) is parameterization of the pad S, then the couple $(r(V),r^{-1})$ is a two-dimensional local map on S, whose area overlaps with parameterization area.

Definition 2. *For two p-dimensional local maps (U,φ) and (U,ψ), on topological space M, we say that they are* ***smoothly connected,***
if the mappings are

$$\psi\circ\varphi^{-1}:\ \varphi(U\cap V)\to\psi(U\cap V),$$
$$\varphi\circ\psi^{-1}:\ \psi(U\cap V)\to\varphi(U\cap V)$$

smooth.

Definition 3. *Let's say that (U, φ) is p—dimensional local map on topological space M. Local coordinates of the point a, in relation to local map (U,φ) we call coordinates of the point $\varphi(a) \in R^n$. Those coordinates will be marked in the following way:*
$(x^1(a),x^2(a),\dots,x^n(a))$.

Let's say that *U* and *V* are the areas of two *n*—dimensional maps on topological space *M*, where $U\cap V \neq \emptyset$. If the point is $a\in U\cap V$, then mapping $\psi\circ\varphi^{-1}$ turns the point $\varphi(a) = (x^1(a),x^2(a),\dots,x^n(a))$, into the point $\psi(a)= (y^1(a),y^2(a),\dots,y^n(a))$, where $y^1,\dots,y^n$ are the functions of *n* variables $x^1,\dots,x^n$, i.e.
$y^i = f^i(x^1,\dots, x^n),\ 1<i<n$. Functions $y^1,\dots,y^n$, i.e. functions,
$y^i = f^i(x^1,\dots, x^n),\ 1<i<n$,
are smooth, if the maps (U, φ) and (V, ψ) are smoothly connected.

Theorem 1. *Two local maps in S, which are determined by trajectories, are smoothly connected.*

The proof of this theorem follows from the proofs of well-known theorem on replacement of parameterization of pad S into R^3.

Definition 4. Smooth n—**dimensional atlas** or **differentiable structure** *on topological space M is the family* $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ *n—dimensional local maps on M, which satisfy the following conditions:*

Any two maps of atlas are smoothly connected;
$\bigcup_{\alpha\in A} U_\alpha = M$ *i.e. areas of all local maps cover M.*

Definition 5. *For smooth atlas, on topological space M, we say that it is* **maximal,** *if any local map on M is smoothly connected with all maps of atlas, and it also belongs to the same atlas.*

Each smooth atlas on *M* can unambiguously be complemented to maximal atlas, by adding all local maps that are smoothly connected with all maps of the atlas.

Definition 6. *Hausdorff's topological space M, along with countable base and n-dimensional atlas which is default in it, is called n-dimensional multiplicity.*

Very often, instead of the term *smooth n-dimensional multiplicity,* we use the term: *n-dimensional differentiable multiplicity.*

For example, if S is a pad in R^3, then each point $a \in S$ belongs to the area of some local map on S, and therefore any two such local maps are smoothly connected. If in the pad S there is a smooth atlas, then we can, if necessary, complement it to the maximal atlas on S. As topological space R^3 disposes with a countable base, S has the same feature as topological subspace. Therefore, any pad S in R^3 is a two-dimensional multiplicity.

As a second example, we take projective pad $P^2(R)$. In order to turn topological space into multiplicity, it is sufficient to construct a smooth atlas on it.

Let's introduce the equivalence relation in the space $R^3 \setminus \{0\} = R_0^3$, by using the formula

$x \sim y \Leftrightarrow y = tx,\ t\epsilon R.$

In this example, equivalence classes are lines that go through coordinate start, where, on them, the coordinate start itself is cut out. Class of equivalence, which contains an arbitrary point $x \in R_0^3$, will be marked as *[x]*. Based on the above, we have

$x \sim y \Leftrightarrow\ [x] = [y].$

Now, let's observe the set whose elements are the introduced classes of equivalence and which are called *projective surface* marked as $P^2(R)$.

Mapping π: $R_0^3 \to P^2(R) : x \to [x]$

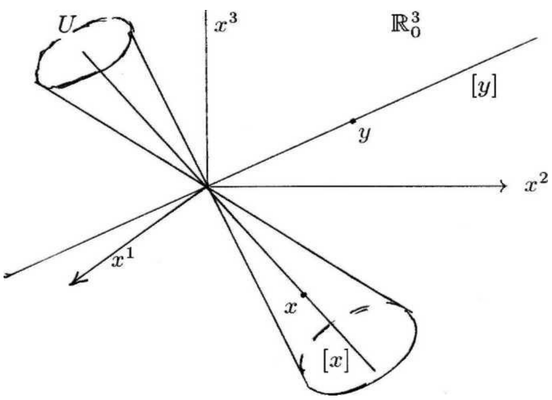

Fig 1

Is a surjection and it will be used to introduce topology into $P^2(R)$. Subset $U \subset P^2(R)$ is open if $\pi^{-1}(U)$ is an open subset in R_0^3. In order words, subset *U* whose elements are lines, is open, if the subset is made of all the points of lines from *U* open in $\mathbf{R_0}^3$ (Figure 1.).

For example, if from the subset *U* iz $P^2(R)$ we throw out all the points, which as lines lie in the surface and go through coordinate start, then the complement of this subset to the subset *U* is an open subset. It is easy to verify whether such formed open subsets in $P^2(R)$ satisfy all axioms of topological space, and if the set $\pi^{-1}(U)$ is open, when the set *U* is open, then the mapping

π: $\mathbf{R_0}^3 \to P^2(R)$

is continuous.

II. THEOREM ON OPEN MAPPING

Theorem 2. *Mapping*
π: $\mathbf{R_0}^3 \to P^2(R)$
is open, i.e. image $\boldsymbol{\pi(W)}$ *of each open subset* $W \subset \mathbf{R_0}^3$ *is open subset in* $P^2(R)$.

Proof. Let's say that $W \subset P^2(R)$ is an open subset in $P^2(R)$. Set $\boldsymbol{\pi(W)}$ is open if and only if $\boldsymbol{\pi^{-1}(\pi(W))}$ is open in $\mathbf{R_0}^3$. Let's consider, for each

$t \in R$, $(t \neq \emptyset)$, the mapping

Φ_t: $\mathbf{R_0}^3 \to \mathbf{R_0}^3$: $\boldsymbol{x \to tx}$,

which is a homomorphism. Then the set $\boldsymbol{\pi^{-1}(\pi(W))}$ can be represented in the form of the union

$$\bigcup_{t\neq 0} \varphi_t(W)$$

of open subsets $\varphi_t(W)$.

Indeed, from a series of equivalences

$x \in \boldsymbol{\pi^{-1}(\pi(W))} \Leftrightarrow \boldsymbol{\pi}(x) \in \boldsymbol{\pi(W)}$

$\Leftrightarrow (\exists y \in W)(\boldsymbol{\pi}(x) = \boldsymbol{\pi}(y)),$

$\Leftrightarrow (\exists t \in R,\ t \neq 0)(x = tx),$

$\Leftrightarrow (\exists t \in R,\ t \neq 0)(x = \varphi_t(y)),$

$\Leftrightarrow (\exists t \in R,\ t \neq 0)(x \in \varphi_t(W)),$

we obtain

$$x \in \bigcup_{t\neq 0} \varphi_t(W)$$

In the end, if each set $\varphi_t(W)$ is open, then the set $\boldsymbol{\pi^{-1}(\pi(W))}$ is open too.

As in the space $\mathbf{R_0}^3$ there is a countable base $\{W_N\}_{\pi} \in N$, and mapping $\boldsymbol{\pi}$ is open and surjective, it follows that $\{W_N\}_{\pi} \in N$ is countable base in $R^2(R)$ as well.

Theorem 3. ***Space*** $P^2(R)$ ***is Hausdorff space.***

Proof. Let's say that $\boldsymbol{[x]}$ and $\boldsymbol{[y]}$ are two different points from $P^2(R)$. By interpretation of points $[\boldsymbol{x}]$ and $\boldsymbol{[y]}$ as two lines from $P^2(R)$, we can construct two real circular cones, without common points $\boldsymbol{W_1}$ and W_2 in $P^2(R)$, with peaks in coordinate start, whose axes are $\boldsymbol{[x]}$ and $\boldsymbol{[y]}$ respectively. In that case, sets $\boldsymbol{\pi}(\boldsymbol{W_1})$ and $\boldsymbol{\pi}(W_2)$ are disjunctive environments of the points $\boldsymbol{[x]}$ and $\boldsymbol{[y]}$ in $P^2(R)$, which means that $P^2(R)$ is Hausdorff space.

III. CONSTRUCTION OF SMOOTH ATLAS

Now, let's construct a smooth atlas in $P^2(R)$. For that purpose, let's mark the coordinates in the space R^3 with $\boldsymbol{(x^1, x^2, x^3)}$. Let's observe the open subsets $W_1 = \{(x^1, x^2, x^3) \in R^3 : x^{\mathbf{i}} \neq 0\}$ $(\mathbf{i} = 1,2,3)$ from $P^2(R)$ whose images $U_{\mathbf{i}} = \boldsymbol{\pi}(W_i)$ are open sets in $P^2(R)$.

We construct the mappings

$\boldsymbol{\varphi}_{\mathrm{i}} : U_{\mathrm{i}} \to R^2$

by formulas

$\varphi_1([(x^1, x^2, x^3)]) = \left(\frac{x^2}{x^1}, \frac{x^3}{x^1}\right),$

$\varphi_2([(x^1, x^2, x^3)]) = \left(\frac{x^1}{x^2}, \frac{x^3}{x^2}\right)$

$\varphi_3([(x^1, x^2, x^3)]) = \left(\frac{x^1}{x^3}, \frac{x^2}{x^3}\right)$

where (x^1, x^2, x^3) is the representative of the class $[x]$. If in formulas $\varphi_1([(x^1, x^2, x^3)])$ we replace argument x with an argument tx, then they do not change, which means that mappings of $\boldsymbol{\varphi}_{\mathbf{i}}$ are

properly defined. Let's consider more carefully one of these mappings, for example, the mapping
φ_3: $U_3 \rightarrow R^2$

Firstly, we must mention that U_3 is a set of lines in space $P^2(R)$ which lie in plane $x^3 = 0$. Let's select a representative of this class on the line $[x] \in U_3$, i.e. let's say that class representative is (x^1, x^2, x^3). Then
$\varphi_3([x]) = (x^1, x^2)$
and $\varphi_3([x])$ represents the first two coordinates of the intersection point of the line $[x]$ and plane $x^3 = 1$ (Figure 2). As the line $[x] \in U_3$ and plane $x^3 = 1$ have only one common point $(x^1, x^2, 1)$**,** and that means that line $[(x^1, x^2, 1)]$ penetrates the plane $x^3 = 1$ in the point $(x^1, x^2, 1)$, and that plane $x^3 = 1$ cuts the line $[(x^1, x^2, 1)]$ in the same point $(x^1, x^2, 1)$. It follows that mapping

φ_i: $U_3 \rightarrow R^2$

is a bijection.

For each open subset $U \subset U_3$, the subset $\pi^{-1}(U)$ is open, and the intersection of $\pi^{-1}(U)$ with the plane $x^3 = 1$ is an open subset in topology of that plane, and open subset is a projection of this intersection φ3(0) on the plane *x3* = 0. Therefore, mapping of φ3 is open. In an analog manner, it is also shown that mapping (φ -1) is open, i.e. that mapping

$\boldsymbol{\varphi_3}$***: $U_3 \rightarrow R^2$,***

is a homomorphism, and thus the couple (U_3, φ_3) is a local map in $P^2(R)$. In that way, we prove that (U_1, φ_1) and (U_2, φ_2) are local maps in $P^2(R)$.

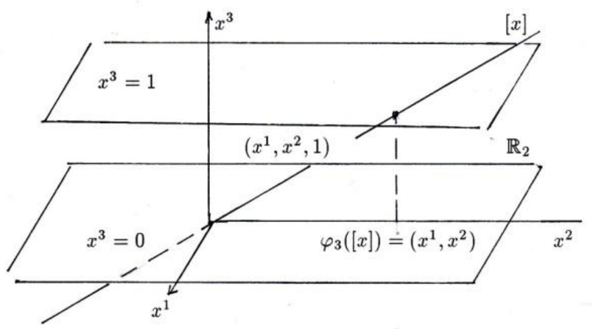

Figure 2.

Definition 7. ***Constructed maps (U_1, φ_1) are called affine maps in projective plane $P^2(R)$.***

It is obvious that areas of affine maps cover entire projective plane. It left to prove that arbitrary two affine maps are smoothly connected. Let's prove this feature, for example, for the maps (u_1, φ_1) and (u_2, φ_2) while the for the rest of the maps it is proved in analog manner.

Indeed, from

$$\varphi_1([(x)]) = \left(\frac{x^2}{x^1}, \frac{x^3}{x^1}\right),$$

we obtain
$\varphi_1^{-1}((u^1, u^2)) = [(1, u^1, u^2)]$.
In an analog manner, we obtain

$$\boldsymbol{\varphi_2} \circ \varphi_1^{-1}((u^1, u^2)) = \boldsymbol{\Phi_2}[(1, u^1, u^2)] = \left(\frac{1}{u^1}, \frac{u^2}{u^1}\right),$$

as well as

$$\boldsymbol{\varphi_2} \circ \varphi_1^{-1}((v^1, v^2)) = \boldsymbol{\Phi_2}[(v^1, 1, u^2)] = \left(\frac{1}{u^1}, \frac{u^2}{u^1}\right).$$

Therefore, mappings $\boldsymbol{\varphi_2} \circ \varphi_1^{-1}$ *and* $\boldsymbol{\varphi_1} \circ \varphi_2^{-1}$ *are smooth, and therefore the maps* **(U_1, φ_1) i (U_2, φ_2)** *are smoothly connected.*

Completely analogous, starting from the space R^{n+1} we can construct a projective space $P^n(R)$, which can therefore be translated into n—dimensional multiplicity.

For example, real line R, with metrics defined on it $d(x, y) = |x - y|$, is a metric space. When we talk about topology on the line R, we usually imply topology induced by default metrics. Open sets in that topology are the unions of open intervals. On the line R, there is a local map that is simultaneously a smooth atlas. That map consists of an open set R and identical homomorphism

$$Id: \ R \rightarrow R: \ t \rightarrow t,$$

which is easily verified if in one case R is observed as a set of points and in the other – as a set of numbers.

If the atlas {(R, Id)} is complemented up to the maximal atlas, then the line R is translated into one-dimensional multiplicity.

Now, we shall describe how each open subset of n-dimensional multiplicity can be translated into n-dimensional multiplicity.

Let's say that M is n—dimensional multiplicity with maximal smooth atlas $\{(U_\alpha, \varphi_\alpha)\}\alpha \in A$ and V open subset in M. Topology of the subset V consists of all open subsets from M that are contained in V. Topological space V is Hausdorff space and it disposes with a countable base. Let's select in the atlas $\{(U_\alpha, \varphi_\alpha)\}\alpha \in A$ a subfamily $\{(U'_\beta, \varphi'_\beta)\}_\beta \in B$ $(V \subset A)$ of all those maps for which $U'_\beta \subset V$. It is obvious that couples $(U'_\beta, \varphi'_\beta)$ of the local map are on **V,** *and that each pair of these maps is smoothly connected, as maps in M. Equation*

$$\bigcup_\beta U'_\beta = V$$

can be verified in the following manner: if $(U_\alpha, \varphi_\alpha)$ is a local map on M and if $(U_\alpha \cap V, \neq \emptyset)$, then $(U_\alpha \cap V, \varphi_\alpha|_{U_\alpha \cap V})$ is a local map on both M and V.

As each point from V belongs to the area of some local map M, it also belongs to the area of local map on V. It is only left to verify whether the constructed atlas is maximal. Let's say that (U', φ') is an arbitrary local map on V, which is smoothly connected to all local maps $(U'_\beta, \varphi'_\beta)$, $\beta \in V$; then (U', φ') is a local map on M and it is smoothly connected to all maps $(U_\alpha, \varphi_\alpha)$, $\alpha \in A$. Due to the maximum of atlas on M, the map selected belongs to maximal atlas and analogous to this, it also belongs to the constructed atlas on *V*.

Subset *V*, along with constructed atlas is a p-dimensional multiplicity and it is called open submultiplicity in M.

IV. CONCLUSION

Multiplicity is a significant term, which is used in many parts of contemporary mathematics. The term multiplicity enables a detailed study of hypersurface in Rn and determination of the curvature of hypersurface and the curve on hypersurfaces. If we observe the curve as a path of the particle in certain time interval, we can determine the length of that path, describe the direction of movement and position of body in space. Acceleration of the body is related with the curvature of the path: higher the curvature, higher the acceleration.

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