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TOPOLOGICAL MODELS FOR THE STUDY OF KINEMATICS OF MULTIBODY SYSTEMS

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Abstract. The paper aims to provide a structural description of flat mechanics by using the theory of graphs. Besides this description, fully defined by the incidence matrix (or by the equivalent basic cycle matrix), geometrical and kinematical description are also intended. The graph theory was used in the mechanical system analysis in two different ways: - in one way the constitutive elements of the system are considered nodes and the liaison between elements are considered edges. This analysis leads to the independent cycle method, developed by various authors in many variants; - another way is to consider the liaison between elements as nodes and the vector defined by the points of the liaison as edges. This description, less used in the literature permit to utilize the natural vector description of the system. The thus created formal framework allows representing the condition equations of speeds and accelerations suitable for numerical applications.

1. INTRODUCTION

The graph theory was used in the mechanical system analysis in two different ways:

- in one way the constitutive elements of the system are considered nodes and the liaison between elements are considered edges[4], [7]. This treatment is equivalent with the Kirchhoff equations for the electrical systems. This analysis leads to the independent cycle method, developed by various authors in many variants. Some descriptions [3], [5] continue this case.
- another way is to consider the liaison between elements as nodes and the vector defined by the points of the liaison as edges. This description, less used in the literature [6] permit to utilize the natural vector description of the system. In the paper is analysed this treatment using a topological approach. The planar case, presented in the paper, can be extended to the spatial systems.

2. TOPOLOGICAL DESCRIPTION OF THE MECHANISM

2.1. Kinematical Equivalent Mechanism

In order to present the method we must to attach a graph to mechanism. The first step is to build a mechanical system made only by bars equivalent, from mechanical point of view, with the system done. We will analyze the planar mechanisms; the developed methods could be applied to other mechanical systems.

It is considered the mechanism presented in fig.1. The liaison between two elements it is accomplished by joints and slide bars.

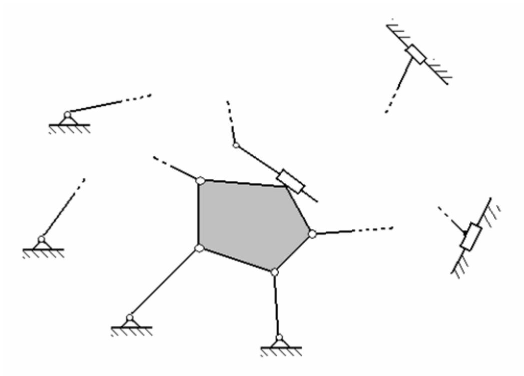

Fig.1

We build a mechanism kinematical equivalent with the mechanism defined like:

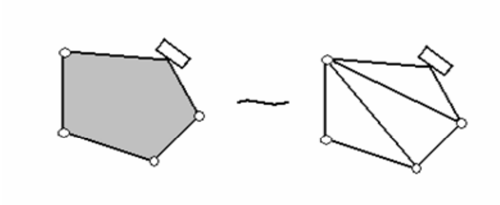

Fig.2

I. We replace any mobile element type plate (fig.2a) with a rigid structure from fig. (2b), formed with triangles made by bars. It is introduced in this way the equations which describe the kinematical state of a fundamental cycles system (independent cycles) “false”, but what they facilitate the equations representation.

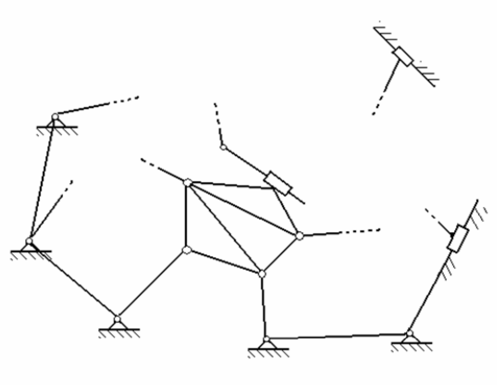

Fig.3

II. It bonds the points that accomplish the bonding with the fixed element through bars type elements that close the polygon (fig.3) and then it is eliminated one of these elements (it results from vectors sum with changed sign of all other elements). It is introduced like this the “false” elements which can modify the length, but not the position, so always angular speed and angular acceleration of such element are null.

By these two constructions it obtains an equivalent kinematical mechanism with the initial mechanism, but made it only with bars (fig.3). In the moment of establishing the kinematical behavior of the mechanism elements, we consider that knowing all the bars length from the equivalent kinematical mechanism and the angles made of them with Ox axis of a xOy system.

Putting to each bar element a vector with the initial point into a liaison and the end point into other liaison, we can build a graph which has, like nodes, the joints and slide of the mechanism, and like edges the mechanism bars, the sense of each line being determined by the direction of the vector attached to correspondent bar. (fig.4).

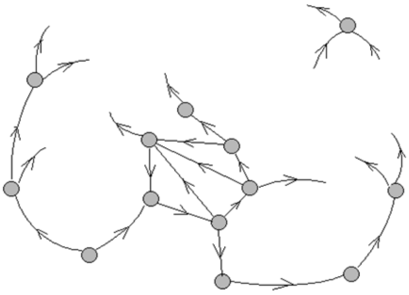

Fig.4

2.2. Independent Cycles

To understand the presentation we summarise some basic notions in graph theory[1],[2]. Being *r+1* the number of the nodes and *s* the number of the edges of the obtained graph. The complete incidence matrix is the matrix $[A_a]=[a_{ij}]$ having the dimension *(r+1) x s* with:

$a_{ij}= 1$ if the edge j is incident to the node i and go out from node ;
$a_{ij}= -1$ if the edge j is incident to the node i and go in to node ;
$a_{ij}=0$ if the edge j is not incident to the node i.

The matrix $[A]$ obtained by eliminating a line in $[A_a]$ is the incidence matrix (or reduced incidence matrix) .

The selection of a tree into the graph allows a re-ordination and a partition of the $[A]$:

$$[A]=[A_t \vdots A_l]$$

following the edge of the tree and the branches (if we attach the branches to tree we reconstruct the initial graph).

The complete cycles matrix is the matrix: $B_a=[b_{ij}]$ with:

$b_{ij}=1$ if the edge *j* makes part from cycle *i* and its orientation coincide with the choice sense for passing the cycle;
$b_{ij}=-1$ if the edge *j* makes part from cycle *i* and its orientation is in contrary sense with the choice sense for passing the cycle;
$b_{ij}=0$ if the edge *j* makes not part from the cycle *i*.

The rank of the matrix B_a is *s-r* . The matrix B_f with rank *s-r* obtained from B_a by retaining of *s-r* independent lines is the cycles fundamental matrix. Between B_f and A there is the following relation: $B_f=[[-A_t^{-1}A_l]^T \vdots I]$

We exemplify for the mechanism of fig. (5):

Mechanism   Kinematical equivalent mechanism   Associated graph

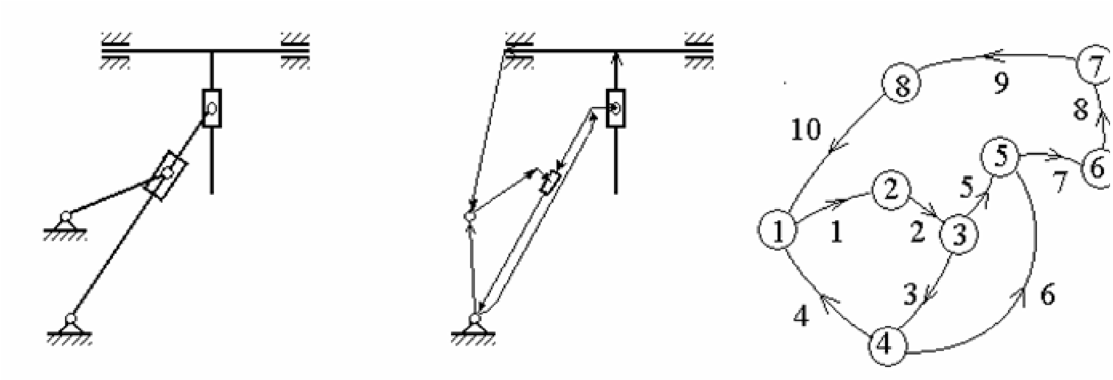

Fig.5

$$
\begin{array}{cc}
 & \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\
\begin{array}{c} 2 \\ 3 \\ 4 \\ [A]=5 \\ 6 \\ 7 \\ 8 \end{array} &
\begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1
\end{bmatrix} = [[A_t][A_l]]
\end{array}
$$

(Elem./Nod)

After calculation, it results the fundamental cycle matrix, like:

$$
[B_f]=\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1
\end{bmatrix}
$$

In this example, to make analyze we have made the replacement from fig.(6).

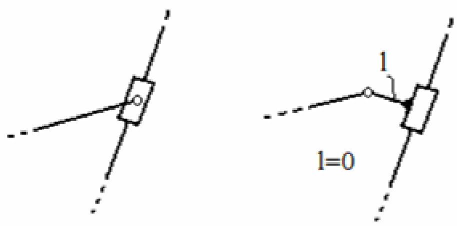

Fig.6

3. KINEMATICAL DESCRIPTION

3.1. Notations .

Being an assembly of *s* values $(e_1, e_2, \ldots, e_s)$. With this we can build the diagonal matrix:

$$[E] = \begin{bmatrix} e_1 & & & & \\ & e_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & e_s \end{bmatrix} \text{ and the vector } \{E\} = \begin{Bmatrix} e_1 \\ e_2 \\ \vdots \\ \vdots \\ e_s \end{Bmatrix}.$$

Being $(l_1, l_2, \ldots, l_s)$ the length of the equivalent mechanism bars. In the presented manner we build the diagonal matrix $[L]$and the vector $\{L\}$.

If $\theta_1, \theta_2, \ldots, \theta_s$ are the angles made by the vectors associated with the bars with Ox axis, we build with $(\cos\theta_1, \cos\theta_2, \ldots, \cos\theta_s)$ the matrix $[C]$ and the vector $\{C\}$, then with $(\sin\theta_1, \sin\theta_2, \ldots, \sin\theta_s)$ the matrix $[S]$ and the vector $\{S\}$. In the same way if we make the notation $\omega_i = \frac{d\theta_i}{dt}$, $v_i = \frac{dl_i}{dt}$, $\varepsilon_i = \frac{d\omega_i}{dt}$, $a_i = \frac{dv_i}{dt}$ we obtain the matrices $[\omega]$, $[v]$, $[\varepsilon]$, $[a]$ and the vectors $\{\omega\}$, $\{v\}$, $\{\varepsilon\}$, $\{a\}$. Being:

$$[F] = \begin{bmatrix} f_1 & & & & \\ & f_2 & & 0 & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & f_s \end{bmatrix} \text{ şi } \{F\} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ f_s \end{Bmatrix}$$

it easy to verify:

$$[F][E] = [E][F] \quad (2)$$

$$\{F\}^T [E] = \{E\}^T [F] \quad (3)$$

Being e_i and f_i variables with time. We have made the notations:

$$[\dot{E}] = \frac{d[E]}{dt} \quad ; \quad \{\dot{E}\} = \frac{d\{E\}}{dt} \quad ; \quad [\dot{F}] = \frac{d[F]}{dt} \quad ; \quad \{\dot{F}\} = \frac{d\{F\}}{dt}$$

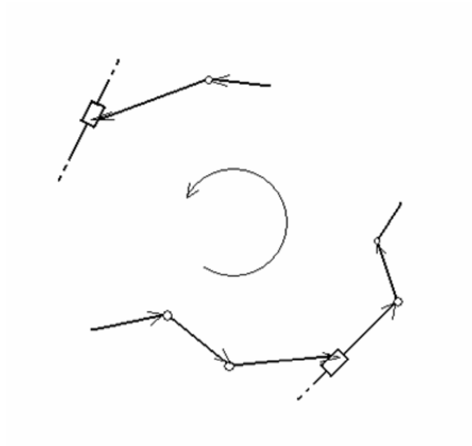

Fig.7

The following relations can be verified:

$$\frac{d\left(\{F\}^T[E]\right)}{dt}=\frac{d\left(\{E\}^T[F]\right)}{dt}= \qquad (4)$$
$$=\{\dot{F}\}^T[E]+\{F\}^T[\dot{E}]=\{\dot{E}\}^T[F]+\{E\}^T[\dot{F}]$$

3.2. Cycle Equations

We consider the cycle *p* formed by the edges *a*,*b*,.. .,*l* (fig.7). To the cycle *p* it corresponds the line p in fundamental cycle matrix named with B_p.

$$\begin{array}{l} Elem.\ 1 \quad 2 \quad \ldots \quad a \quad \ldots \quad b \quad \ldots \quad l \quad \ldots \quad s \\ B_p=\begin{bmatrix}0 & 0 & \ldots & 0 & a & 0 & \ldots & 0 & b & 0 & \ldots & 0 & l & 0 & \ldots & s\end{bmatrix}.\end{array}$$

The contour equation for this cycle will be:

$$\vec{l}_a+\vec{l}_b+\ldots+\vec{l}_l=0$$

or, projecting on the two axis:

$$\begin{aligned} l_a\cos\theta_a+l_b\cos\theta_b+\ldots\ldots+l_l\cos\theta_l=0 \\ l_a\sin\theta_a+l_b\sin\theta_b+\ldots\ldots+l_l\sin\theta_l=0 \end{aligned} \qquad (5)$$

Being $\{L_x\}$ the vector $\{l_1\cos\theta_1 \quad l_2\cos\theta_2 \quad \ldots \quad l_s\cos\theta_s\}^T$ and $\{L_y\}$ the vector $\{l_1\sin\theta_1 \quad l_2\sin\theta_2 \quad \ldots \quad l_s\sin\theta_s\}^T$, the contour equation for the cycle *p* will write:

$$\lfloor B_p\rfloor\{L_x\}=\{0\} \quad ; \quad \lfloor B_p\rfloor\{L_y\}=\{0\}. \qquad (6)$$

Making $p=1,2,\ldots,r-s$ and because:

$$[B_f]=\begin{bmatrix}B_1\\B_2\\\vdots\\B_{r-s}\end{bmatrix}.$$

the contour equations for the entire mechanism could write:

$$\lfloor B_f\rfloor\{L_x\}=\{0\} \quad ; \quad \lfloor B_f\rfloor\{L_y\}=\{0\}.$$

We can observe that:

$$\{L_x\} = [L]\{C\} = [C]\{L\} \quad ; \quad \{L_y\} = [L]\{S\} = [S]\{L\}.$$

3.3. Constraint relations in terms of velocities and accelerations

Because:

$$[\dot{C}] = -[S][\omega] \quad ; \quad [\dot{S}] = [C][\omega] \quad ; \quad [\dot{L}] = [v] \quad ,$$

we have, due to the relation (4):

$$\{\dot{L}_x\} = -[L][S]\{\omega\} + [C]\{v\} \quad ; \quad \{\dot{L}_y\} = [L][C]\{\omega\} + [S]\{v\} \ . \tag{7}$$

We can write the conditions equations for the velocities obtained by differentiation of the relations (6):

$$[B_f]\{\dot{L}_x\} = \{0\} \quad ; \quad [B_f]\{\dot{L}_y\} = \{0\}. \tag{8}$$

We have then:

$$\begin{cases} \{\ddot{L}\}_x = -[L][C][\omega]\{\omega\} - [L][S]\{\varepsilon\} - 2[S][v]\{\omega\} + [C]\{a\} \\ \{\ddot{L}\}_y = -[L][S][\omega]\{\omega\} + [L][C]\{\varepsilon\} - 2[C][v]\{\omega\} + [S]\{a\} \end{cases} \tag{9}$$

The differentiation of the relations (8) allows the writing of the conditions equations for the accelerations:

$$[B_f]\{\ddot{L}_x\} = \{0\} \quad ; \quad [B_f]\{\ddot{L}_y\} = \{0\} \ . \tag{10}$$

3.4. Kinematical Analysis

The number of the equations establish by the relation (8) is *2* x *(s-r)*, so it can determinate *2* x *(s-r)* unknown – angular and linear velocities. It results that for determination of the mechanism kinematics must know the other *s-2* x *(s-1)* values, velocities that appear in relations (8).

The considerations remain valuable also for the accelerations (relations (10)).

If two elements i and j are connected between them through a sliding bar (fig.8) results $\omega_i = \omega_j$. This connection allows the reduction of the angular speeds number from the vector $\{\omega\}$. It can write so:

$$\{\omega\} = [R_2]\{\omega^*\} \tag{11}$$

where $\{\omega^*\}$ contain independent angular velocities.

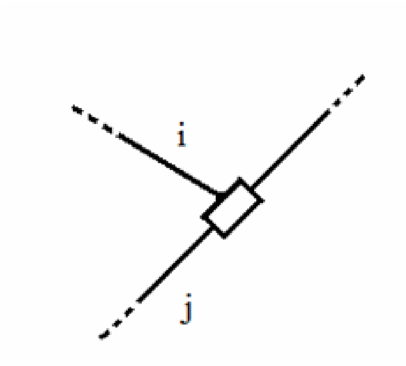

Fig.8

If the element i has two joints on the both sides or a joint and a sliding bar rigid tied (fig.9) then:

$$\frac{dl_i}{dt}=0 \quad .$$

In the vector $\{v\}$ there is a lot of null elements. We make the notation $\{v^*\}$ for the no-null components vector. We have:

$$\{v\}=[R_1]\{v^*\} \; . \tag{11'}$$

The relations (11) , (11') remain valuable also for the accelerations:

$$\{\varepsilon\}=[R_1]\{\varepsilon^*\} \tag{12}$$

$$\{a\}=[R_1]\{a^*\} \; . \tag{12'}$$

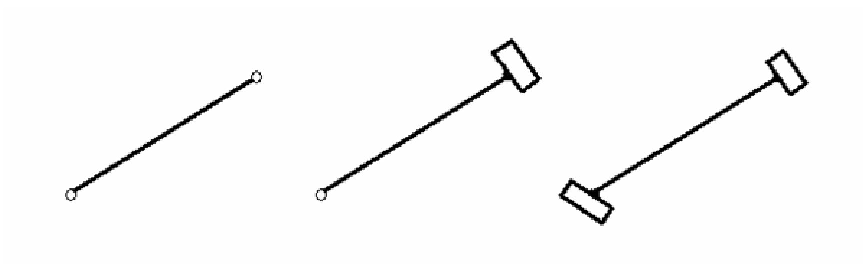

Fig.9

The relations (7) become:

$$\{\dot{L}_x\}=-[L][S][R_2]\{\omega^*\}+[C][R_1]\{v^*\}=\left[-[L][S][R_2] \quad [C][R_1]\right]\begin{Bmatrix}\omega^*\\ v^*\end{Bmatrix} \quad ;$$

$$\{\dot{L}_y\}=[L][C][R_2]\{\omega^*\}+[S][R_1]\{v^*\}=\left[[L][C][R_2] \quad [S][R_1]\right]\begin{Bmatrix}\omega^*\\ v^*\end{Bmatrix} \quad ,$$

so the condition equations for velocities (8) become:

$$\begin{bmatrix}-[B_f][L][S][R_2] & [B_f][C][R_1]\\ -[B_f][L][C][R_2] & [B_f][S][R_1]\end{bmatrix}\begin{Bmatrix}\omega^*\\ v^*\end{Bmatrix}=0 \tag{8'}$$

We re-ordinate the vector lines $\begin{Bmatrix}\omega^*\\ v^*\end{Bmatrix}$ in such way that on the first *2* x *(s-r)* positions could be the unknown values and we make the notation:

$$\{\dot{q}\}=[\dot{q}_1\ \dot{q}_2 \dots \dot{q}_{2x(s-r)}\ \dot{q}_{2x(s-r)+1} \dots \dot{q}_1]^T=\begin{Bmatrix}\{\dot{q}\}_n\\ \cdots\\ \{\dot{q}\}_c\end{Bmatrix}$$

where $\dot{q}_i$ is an element from ω^* or v^*, i=1,2,... 2x*(s-r)* that indicate the unknown elements, *i=2*x*(s-r)+1*,... ,*s* that indicate known elements. Making the re-ordination of the matrix columns:

$$\begin{bmatrix}-[B_f][L][S][R_2] & [B_f][C][R_1]\\ -[B_f][L][C][R_2] & [B_f][S][R_1]\end{bmatrix}$$

that correspond to the re-ordination through $\begin{Bmatrix}\omega^*\\ v^*\end{Bmatrix}$ passing in $\{\dot{q}\}$ and making the notation of this matrix after re-ordination with $[P]$, the relations (8') could be write:

$$[P]\{\dot{q}\}=0 \tag{8”}$$

Partitioning $[P]=[P_n\ P_c]$ in a way that the multiplying the partitioned matrix:

$$[P_n\ P_c]\begin{Bmatrix}\{\dot{q}\}_n\\ \{\dot{q}\}_c\end{Bmatrix}=0$$

could have sense, the condition equations for the velocities become:

$$[P]_n\{\dot{q}\}_n+[P]_c\{\dot{q}\}_c=0 \tag{8’’’}$$

From here it can be possible to establish the vector of the unknown velocities:

$$\{\dot{q}\}_n=-[P]_n^{-1}[P]_c\{\dot{q}\}_c \tag{13}$$

Now, it is easy to build the vectors:

$$\{\omega\}^T[\omega]=[\omega_1^2\ \ \omega_2^2\ \ \dots\ \ \omega_s^2]^T$$

and:

$$\{v\}^T[\omega]=[v_1\omega_1\ \ v_2\omega_2^2\ \ \dots\ \ v_s\omega_s^2]^T$$

Making the notations:

$$\{q\}_2=\{\omega\}^T[\omega]\qquad;\qquad\{q\}_{cc}=\{v\}^T[\omega]$$

With these notations, the relations (9) become :

$$\{\ddot{L}_x\}=-[L][C]\{q\}_2-[L][S][R_2]\{\varepsilon^*\}-2[S]\{q\}_{cc}+[C][R_1]\{a^*\}$$

$$\{\ddot{L}_y\}=-[L][S]\{q\}_2+[L][C][R_2]\{\varepsilon^*\}-2[C]\{q\}_{cc}+[S][R_1]\{a^*\}$$

and the conditions (10) become:

$$\begin{bmatrix}-[B_f][L][S][R_2] & [B_f][C][R_1]\\ -[B_f][L][C][R_2] & [B_f][S][R_1]\end{bmatrix}\begin{Bmatrix}\varepsilon^*\\ a^*\end{Bmatrix}=\begin{bmatrix}[B_f][L][C]\\ [B_f][L][S]\end{bmatrix}\{q\}_2+2\begin{bmatrix}[B_f][S]\\ [B_f][C]\end{bmatrix}\{q\}_{cc}$$

(10’)

Making the notations:

$$[Q_2]=\begin{bmatrix}[B_f][L][C]\\ [B_f][L][S]\end{bmatrix}\ ;\ [Q_c]=\begin{bmatrix}[B_f][S]\\ [B_f][C]\end{bmatrix}\ ;$$

$$\{\ddot{q}\}=[\ddot{q}_1\ \ddot{q}_2\ \dots\ \ddot{q}_{2x(s-r)}\ \ddot{q}_{2x(s-r)+1}\ \dots\ \ddot{q}_1]^T=\begin{Bmatrix}\{\ddot{q}\}_n\\ \dots\\ \{\ddot{q}\}_c\end{Bmatrix}$$

we obtain:

$$[P]\{\ddot{q}\}=[Q_2]\{q\}_2+2[Q_c]\{q\}_c \tag{10”}$$

or:

$$[P]_n\{\ddot{q}\}_n+[P]_c\{\ddot{q}\}_c=[Q_2]\{q\}_2+2[Q_c]\{q\}_c \tag{10’’’}$$

and results the unknown accelerations vector:

$$\{\ddot{q}\}_n=-[P]_n^{-1}[P]_c\{\ddot{q}\}_c+[P]_n^{-1}([Q_2]\{q\}_2+2[Q_c]\{q\}_c) \tag{14}$$

The representation of the solutions for velocities and accelerations in the form (13) and (14) allows an approach of the kinematical determination of the planar mechanisms with the computer help.

4. CONCLUSIONS

The presented method offers the kinematical unknowns in the case of the planar mechanism, using the topological properties of the mechanical systems with the associated graph. This method, using the natural description of a mechanical system with the vector algebra leads to an automatic description of such system. It is possible to obtain the velocities and the accelerations. At the positions level the problem is more complicated due to the multiple solutions and the nonlinearity of the equations.

ACKNOWLEDGEMENTS

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